§4.9 Problems: Elementary Properties of Graphs

These are some problems to practice the material above and do not represent homework unless explicitly mentioned otherwise. Give them a try! Some of them will be discussed by your TA during the upcoming discussion sessions from 4 to 5 PM on Tuesdays and Thursdays.

Problem 4.9.1. Prove that there is no simple graph with 8 vertices total, with degrees 2, 2, 3, 3, 3, 4, 4, 4.

Problem 4.9.2. Prove that for any simple graph with n vertices and m edges, there is a vertex with degree at least $2m/n$.

Problem 4.9.3. If H_1, H_2 are connected subgraphs of G, and $H_1 \cap H_2$ is not null, then $H_1 \cup H_2$ is connected.

Problem 4.9.4. Let T be a tree, and let T_1, T_2 be connected subgraphs of T with $V(T_1 \cap T_2) \neq \emptyset$. Show that $T_1 \cup T_2$ and $T_1 \cap T_2$ are trees.

Problem 4.9.5. Prove that every graph $G = (V, E)$ has at least $|V| - |E|$ connected components.

Problem 4.9.6. If G is connected and has no path with length $> k$, then every two paths in G of length k have at least one vertex in common.

Problem 4.9.7. Let v be an arbitrary vertex in a graph G. If every vertex is within distance d of v, then the diameter of the graph is at most $2d$.

Problem 4.9.8. Prove or disprove that if u, v, w are vertices of G, and there is an even length path from u to v and an even length path from v to w, then there is an even length path from u to v.

Problem 4.9.9. In a graph G with n vertices every vertex has degree at least $\frac{n-1}{2}$. Prove that the graph is connected.

Problem 4.9.10. Let G be a graph with $|E(G)| = |V(G)| - 1$. Show that G is connected if and only if G has no cycles.

Problem 4.9.11. Let G be a graph which has a closed walk of odd length. Is it true that G has a cycle of odd length?

Problem 4.9.12. If G is connected and $v \in V(G)$, v is **deletable** if $G - v$ is also connected. Up to isomorphism, how many connected simple graphs G are there with $|V(G)| = 100$ and with at most two deletable vertices?

Problem 4.9.13. If a graph G has average degree $2t$, then it contains every tree T with t edges.

Problem 4.9.14. Let T_1, T_2 be two spanning trees of a graph G, and let $e_1 \in E(T_1) - E(T_2)$. Prove there is an edge $e_2 \in E(T_2) - E(T_1)$ such that e_2 belongs to the fundamental cycle of e_1 with respect to T_2 and e_1 belongs to the fundamental cycle of e_2 with respect to T_1.

Problem 4.9.15. Let G be a connected graph and for each edge e let $w(e)$ be some real number. Let T be a spanning tree of G such that for every edge $f \in E(G) - E(T)$, $w(f) \geq w(e)$ for every edge e of the fundamental cycle of f. Show that T is a min-cost tree.

Problem 4.9.16. Suppose graph G with n vertices contains at least $\frac{4n}{3}$ edges. Then G contains two intersecting cycles.

Problem 4.9.17. Prove that a graph with at most two odd cycles has chromatic number of at most 3.

Problem 4.9.18. Let G be a graph where every two odd circles have at least a vertex in common. Prove that G is 5-colorable.

Problem 4.9.19. Let Q_n denote the graph whose vertex set is $\{0, 1\}^n$ (i.e. there are exactly 2^n, each labeled with a distinct n-bit string), and with an edge between vertices x and y if and only if x and y differ in exactly one coordinate. This is called the n-**dimensional hypercube**.

a) If n is even, prove that Q_n has an Eulerian circuit.

b) In general, prove that Q_n has a Hamiltonian path.

c) Does (b) resemble anything you have seen before?

§4.10 Problems: Planar Graphs

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Problem 4.10.20. Prove that any simple, connected planar graph contains a vertex of degree at most 5.

Problem 4.10.21. Prove that any simple, connected planar graph is 6-colorable.

Problem 4.10.22. Improve the argument from the previous exercise and show that any simple, connected planar graph is 5-colorable. (Hint: You can use Kuratowski's theorem).

Problem 4.10.23. A graph is a *minor* of a graph G if H can be obtained from a subgraph of G by a sequence of edge contractions. Prove that K_5 and $K_{3,3}$ are both minors of the Petersen graph.

Problem 4.10.24. Prove that a simple graph G is a tree if and only if G has no loop as a minor.

Problem 4.10.25. Prove that a graph G is planar if and only if K_5 and $K_{3,3}$ are not minors of G. Note that this is weaker than Kuratowski's theorem, thus prove it from first principles.

Problem 4.10.26. Graph G is **outerplanar** if it can be drawn in the plan so that every vertex is incident with the infinite region. Show that a graph G is outerplanar if and only if G has no K_4 or $K_{2,3}$ minor.

Problem 4.10.27. Prove that if G is planar then G is a minor of a large enough **grid graph**. Formally, an $r \times r$ grid graph is a graph $H(V, E)$, where $V = \{1, \ldots, r\} \times \{1, \ldots, r\}$ and two pairs (i, j) and (i', j') are connected by an edge if and only if $|i - i'| + |j - j'| = 1$. More intuitively, they're just the graphs associated with the geometric square lattice grids.)